Conserved Quantities and the Algebra of Braid Excitations in Quantum Gravity

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February 15, 2008

Abstract

We derive conservation laws from interactions of braid-like excitations of embedded framed spin networks in Quantum Gravity. We also demonstrate that the set of stable braid-like excitations form a noncommutative algebra under braid interaction, in which the set of actively-interacting braids is a subalgebra.

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Contents

1 Introduction

Recently, there has been a significant amount of work done towards a quantum theory of gravity with matter as topological invariants[1, 2, 13, 3, 8, 10, 12, 11]. [1, 2, 13] work on framed three-valent spin networks present in models related to Loop Quantum Gravity with non-zero cosmological constant[5, 6], in which the topological invariants of ribbon braids are able to detect chirality and code chiral conservation laws. However, the results of this approach have a serious limitation in the sense that there is no dynamics of the conserved quantities[2].

To resolve this limitation, a new approach based on embedded four-valent spin networks is proposed in [3], and is shown to have dynamics built in by means of the so-called dual Pachner moves[8]. Here the four-valent spin networks can be understood as those naturally occur in spin foam models[7], or in a more generic context as the original proposal of spin networks put forward by Penrose[9], plus embedding.

The dynamical objects found by the new approach are three-strand braids, each of which is formed by three common edges of two adjacent nodes of the embedded four-valent spin network. The stable three-strand braids, under certain stability condition, are local excitations[8, 14]. Among all stable braids, there is a small class of braids which are able to propagate on the spin network. The propagation of these braids are chiral, in the sense that some braids can only propagate to their left with respect to the local subgraph containing the braids, while some only propagate to their right and some do both[3, 8]. There is another small class of braids, the actively-interacting braids (hereafter called "active braids" for short); each is two-way propagating and is able to merge with its neighboring braid when the interaction condition is met[8]. In the sequel, braids that are not active are called passive, including stationary braids, i.e. those do not even propagate.

[3, 8] are based on a graphic calculus developed therein. However, although the graphic calculus has its own advantages - in particular in describing, e.g. the full procedure of the propagation a braid, it is not very convenient for finding conserved quantities of a braid which are useful to characterize the braid as a matter-like local excitation. In view of this, [10] proposed an algebraic notation of the active braids and derived conserved quantities by means of the new notation.

To these ends, in this paper, we generalize the algebraic notation in[10] to the case of generic three-strand braids. Within this notation, the algebraic equivalence moves are defined and the quantities conserved under these are identified. Finally the algebra of interactions between active braids and passive braids is discussed. This leads to the following results:

1. There exist conserved quantities under interactions and we are able to show the form of these conservation laws.
2. Precise algebraic forms of braid interactions are presented.
3. The set of all stable braids form an algebra under braid interaction, in which the set

of all active braids is the subalgebra.

4. This algebra is noncommutative due to the fact that the left and right interactions of an active braid onto another braid are not the same in general. Conditions of commutative interactions are explicitly given.

5. Asymmetric interactions can be related by discrete transformation, such as P, T, CP, and CT.

An immediate application of these results is realized in a companion paper[11] which discovers the C, P, and T transformations of braids by means of conserved quantities found in [10] and in this paper.

2 Notation

We will extend the algebraic notation of active braids to the general case, namely to propagating braids and in fact all braids. However, for illustrative purposes we keep the graphic notation wherever necessary. We adopt the graphical notation we proposed in [3, 8]. A generic 3-strand braid is shown in Fig. 1(a), while a concrete example is depicted in Fig. 1(b). More precisely, what are shown in Fig. 1 are braid diagrams as projections of the true 3-strand braids embedded in a topological three manifold. Each spin network can be embedded in various ways, some of which are diffeomorphic to each other. The projection of a specific embedding of a braid is called a braid diagram; many braid diagrams are equivalent and belong to the same equivalence class, in the sense that they correspond to the same braid and can be transformed into each other by equivalence moves[3]. Thus a braid refers to the whole equivalence class of its braid diagrams. However, one can choose a braid diagram of an equivalence class as the representative of the class, we therefore will not distinguish a braid from a braid diagram in the sequel unless an ambiguity arises. Besides, a braid always means a 3-strand braid.

It is important to emphasize the choice of the representative of an equivalence class of braid diagrams. In [8], each equivalence class of braid diagrams is represented by its unique element which has zero external twists (see Fig. 1(b) for an example). This choice makes the propagation and interaction of braids defined in[8] easier to handle. However, there are three types of stable braids, viz active braids, propagating braids, and stationary braids[8, 12]. **Propagating** braids are able to exchange places with their adjacent substructures in the graph under the local dynamical moves, whereas the **stationary braids** cannot propagate in the way the propagating braids do. These braids are in most cases represented by braid diagrams of zero external twists.

On the other hand, as pointed out in [8, 10], the **active** braids, each of which can propagate and can interact onto any other braid in the sense that it can merge with another adjacent braid to form a new braid as long as the interaction condition is met, are happen to be both **completely left-** and **right-reducible**, i.e. such a braid is always equivalent to a trivial braid diagram with possibly twists on its three strands and two external edges. Thus

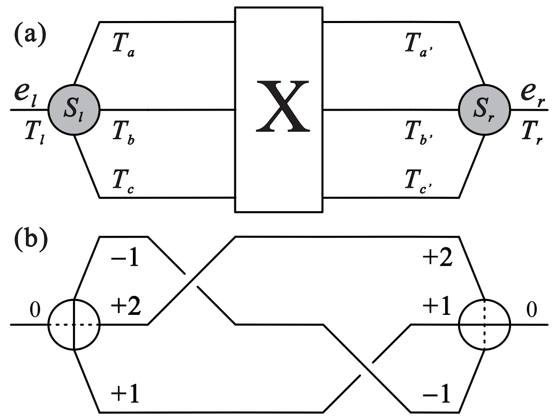

Figure 1: (a) is a generic 3-strand braid diagram formed by the three common edges of two end-nodes. S_l and S_r are the states of the left and right **end-node**s respectively, taking values in $+$ or $-$. X represents a sequence of crossings, from left to right, formed by the three strands between the two nodes. T_a, T_b, and T_c are the **internal twists** respectively on the three strands from top to bottom, on the left of X. T_l and T_r, called **external twists**, are respectively on the two external edges e_l and e_r. All twists are valued in $\mathbb{Z}$ in units of $\pi/3$[3]. (b) is a concrete example of a braid diagram, in which the left end-node is in the $'+'$ state while the right end-node is in the $'-'$ state.

it is more convenient to represent each of these braids by a trivial braid diagram (which is not unique) of the corresponding equivalence class. In fact, [10] chose this representation and derived conserved quantities of this type of braids under interaction by introducing an algebraic notation of them and a symbolic way of handling the interactions of them. This trivial representation of active braids is actually a special case of the so-called extremal representation of generic braids[3]. A braid diagram as an extremal representation of a braid is called an **extremum** of the braid. The name-extremum-manifests per se its meaning: a braid diagram with least number of crossings, among all braid diagrams in the same equivalence class.

Therefore, our generalized algebraic notation of a braid should take care of all possible choices of representative of a braid, especially the aforementioned special representations. Note that, the class of propagating braids excludes any active braid, which is also propagating though.

Let us now concentrate on the generic case depicted in Fig. 1(a). Apart from the internal twists, the interior of a braid, which is the region between the two end-nodes and is characterized by the sequence of crossings, satisfies the definition of an ordinary braid, arranged horizontally. We can thus denote the sequence of crossings X by generators of the braid group B_3. The group B_3 has two generators and their inverses. Since we arrange a braid diagram horizontally, the generator and its inverse formed by the upper two strand of a braid are named u and u^{-1} respectively, while the one and its inverse

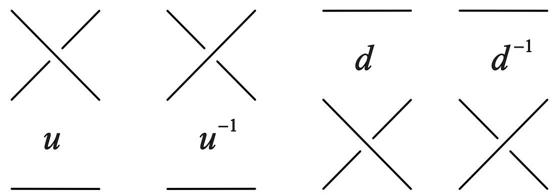

Figure 2: The generators of braid group B_3, and hence of the crossing sequence of a generic 3-strand braid.

formed by the lower two strand of a braid are d and d^{-1} respectively. This convention is illustrated in Fig. 2. Then for example, the crossing sequence in Fig. 1(b) reads $X = u^{-1}d$, from left to right. We also assign an integral value, the crossing number, to each generator, i.e. $u = d = 1$ and $u^{-1} = d^{-1} = -1$.

For an arbitrary sequence X of **order** $n = |X|$, namely the number of crossings, we can write $X = x_1 x_2 \cdots x_i \cdots x_n$, where $x_i \in \{u, u^{-1}, d, d^{-1}\}$ represents the i-th crossing from the left. Therefore, each x_i in X has a two-fold meaning: on the one hand, it is an abstract crossing; on the other hand, it represents an integral value, 1 or -1. When a x_i appears in a multiplication it is usually understood as an abstract crossing, while in a summation it is normally an integer. Note that, as generators of the group B_3, the generators of X obey the following equivalence relations.

$$\begin{aligned} udu^{-1} &= d^{-1}ud \\ u^{-1}du &= dud^{-1} \\ udu &= dud \end{aligned} \tag{1}$$

We assume in any X, the above equivalence relations have been applied to remove any pair of a crossing and its inverse. For example, the sequence $udu^{-1}d^{-1}$ should have been written as $udu^{-1}d^{-1} = d^{-1}udd^{-1} = d^{-1}u$ by the first relation above. The crossing sequence X clearly induces a permutation, denoted by σ_X, of the three strands of a braid. It is obvious that the induced permutation σ_X takes value in S_3, the permutation group of the set of three elements. In terms of disjoint cycles,

$$S_3 = \{\mathbb{1}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}. \tag{2}$$

More precisely, the three internal twists in a triple (T_a, T_b, T_c) on the left of the X are permutated by the induced permutation into $(T_a, T_b, T_c)\sigma_X = (T_{a'}, T_{b'}, T_{c'})$, where $(T_{a'}, T_{b'}, T_{c'})$ is the triple of the internal twists on the right of X. Here σ_X is defined to be a *left-acting* function on the triple of internal twists for two reasons. Firstly, this is a convention of permutation group. Secondly, when there is another crossing sequence, say X', appended to the right of X, which usually happens in interactions of braids, we naturally have $(T_a, T_b, T_c)\sigma_X \sigma_{X'} = (T_a, T_b, T_c)\sigma_{XX'}$, such that the induced permutation of the newly appended crossings is applied after the action of σ_X.

On the other hand, $(T_a, T_b, T_c) = \sigma_X^{-1}(T_{a'}, T_{b'}, T_{c'})$, where σ_X^{-1} is the inverse of σ_X and is a *right-acting* function of the triple. One should keep in mind that the indices of internal twists such as T_a and T_a', a and a' are abstract and have no meaning until their values and positions in the triple of internal twists are fixed. So $(T_a, T_b, T_c) = (T_d, T_e, T_f)$ means respectively $T_a = T_d, T_b = T_e$, and $T_c = T_f$. In the rest of the paper, we will also consider the addition of two triples of twists, i.e. $(T_a, T_b, T_c)+(T_d, T_e, T_f) = (T_a+T_d, T_b+T_e, T_c+T_f)$. Therefore, we can denote a generic braid as Fig. 1(a) by

$$ {}^{S_l}_{T_l}[(T_a, T_b, T_c)\sigma_X]^{S_r}_{T_r} \ , $$

or equally well by

$$ {}^{S_l}_{T_l}[\sigma_X^{-1}(T_{a'}, T_{b'}, T_{c'})]^{S_r}_{T_r} \ . $$

In such a way, which side of the crossing sequence a triple of internal twists is on is transparent. For instance, the braid in Fig. 1(b) can be written as ${}^{+}_{0}[(-1, 2, 1)\sigma_{u^{-1}d}]^{-}_{0}$ or ${}^{+}_{0}[\sigma^{-1}_{u^{-1}d}(2, 1, -1)]^{-}_{0}$, where $\sigma_{u^{-1}d} = (1\ 3\ 2)$ and $\sigma^{-1}_{u^{-1}d} = (1\ 2\ 3)$.

It is now manifest that a generic braid is characterized by the 8-tuple, $\{T_l, S_l, T_a, T_b, T_c, X, S_r, T_r\}$. As mentioned before, S_l and S_r are just signs, $+$ or $-$, such that $-(+) = -$ and $-(-) = +$. Hence, for an arbitrary end-node state S, we may use both $-S$ and $\bar{S}$ for the inverse of S. In fact, this 8-tuple is not completely arbitrary for different type of braids. For a propagating braid B of n crossings represented by the braid diagrams with no external twists we have the following constraints.

1. $T_l = T_r = 0$.

2. The triple (S_l, X, S_r) is not arbitrary. If B is **(left-) right-propagating**, according to [3, 8], B must be **(left-) right-reducible**, in particular its first crossing on the (left) right can be eliminated by the equivalence move, a $\pi/3$ rotation which also flips the (left) right end-node. That is, letting $X = x_1x_2\cdots x_{n-1}x_n$, then under a $\pi/3$ rotation on the (left) right end-node, the triple (S_l, X, S_r) becomes $((\bar{S}_l, x_2\cdots x_{n-1}x_n, S_r))$ $(S_l, x_1x_2\cdots x_{n-1}, \bar{S}_r)$.

3. The triple (T_a, T_b, T_c) is not arbitrary; however, the general pattern of them, ensuring the propagation of B, has not yet been found and is under investigation. But the algebra formulated and the conserved quantities to be found in this paper may turn out to be helpful to resolve this problem.

Any braid whose characterizing 8-tuple violates the above constraints is not propagating

It is useful in certain situation to represent a propagating braid by its **extrema** in the corresponding equivalent class, which are the braid diagrams of the least number of crossings, obtained by rotations of the unique representative with zero external twists, getting rid of the reducible crossings[3]. However, we would like to leave the discussion of this representation to the next section after we defined rotations symbolically.

For an active braid, we choose to represent it by its extrema, i.e. its trivial braid diagrams with external twists. This has actually been carried out in [10], we thus will not

repeat any detail here. It is good to see our notation of the generic case reduces to that of the active braids defined in [10]. For active braids represented by trivial diagrams, the crossing sequence is trivial and hence the induced permutation is the identity, i.e. $\sigma_X = \mathbb{1}$. There is no difference between the triple of internal twists on the left of X and the one on the right. Moreover, we have $S_l = S_r = S$ in this case[10]. As a result, the generic notation uniquely boils down to

$$ {}^{S}_{T_l}[T_a, T_b, T_c]^{S}_{T_r} \ , $$

which is the very notation introduced in [10] for active braids in their trivial representations.

3 Algebra of equivalence moves: symmetries and relations

Since an active braid can always be reduced to trivial braids with twists, it is sufficient to discuss the algebra of simultaneous rotations for these trivial braids. In [10], we have found general effects of simultaneous rotations on trivial braids, especially conserved quantities under this class of equivalence moves. For generic braids, however, we need to consider more generalized rotations of a braid, denoted by $R_{m,n}$ with $m, n \in \mathbb{Z}$, which is the combination of an $m\pi/3$ rotation on the left end-node and an $n\pi/3$ one of the right end-node of the braid. Let us record the algebraic form of such a rotation on a generic braid, and then explain it,

$$ \begin{aligned} &R_{m,n}({}^{S_l}_{T_l}[(T_a, T_b, T_c)\sigma_X]^{S_r}_{T_r}) \\ &={}^{(-)^m S_l}_{T_l+m}[(P^{S_l}_m(T_a - m - n, T_b - m - n, T_c - m - n))\sigma_{X_l(S_l,m)XX_r(S_r,n)}]^{(-)^n S_r}_{T_r+n} . \end{aligned} \tag{3} $$

On the RHS of Eq. 3, the original end-node states, S_l and S_r, become $(-)^m S_l$ and $(-)^n S_r$ respectively. This is because by [3], a $\pi/3$ rotation of a node always flips the state of the node once, which means a $m\pi/3$ rotation should flip the state of a node m times. Also according to [3], a rotation of the left (right) end-node of a braid creates a crossing sequence, appended to the left (right) of the original crossing sequence of the braid. In Eq. 3, the newly-generated sequence on the left is denoted by a function $X_l(S_l, m)$, depending on the original left end-node state and the amount of rotation, m. Likewise, the new crossing sequence on the right is denoted by the function $X_r(S_r, n)$. We will elaborate these two functions shortly. As a consequence, the induced permutation by the crossing sequence changes accordingly, from σ_X to $\sigma_{X_l(S_l,m)XX_r(S_r,n)}$.

In addition, the left triple of internal twists is affected by the rotation of the left end-node, which induces a permutation $P^{S_l}_m$ on the triple, determined by the original end-node state and the amount of rotation m. This function, which obviously takes its value in the group S_3 shown in Eq. 2, is the same as the that induced by a simultaneous rotation on active braids, defined in [10]. One may wonder why the similar permutation induced by the rotation on the right end-node does not appear in Eq. 3. This is due to the advantage of our notation which needs the triple of internal twists on one side, while the triple on

the other side is taken care of by the permutation σ. If one indeed wants to have the triple of internal twists beside the right end-node explicit, one can use the alternative form of the rotation as follows instead.

$$\begin{aligned}
&R_{m,n}({}^{S_l}_{T_l}[\sigma_X^{-1}(T_{a'},T_{b'},T_{c'})]^{S_r}_{T_r})\\
&={}^{(-)^m S_l}_{T_l+m}[\sigma^{-1}_{X_l(S_l,m)XX_r(S_r,n)}(P^{S_r}_{-n}(T_{a'}-m-n,T_{b'}-m-n,T_{c'}-m-n))]^{(-)^n S_r}_{T_r+n},
\end{aligned} \tag{4}$$

Finally, the common increment of $-m-n$ of all internal twists, and the changes of the two external twists under the rotation $R_{m,n}$ in Eq. 3 and Eq. 4 are simple effects of the rotation[3].

We now explain more about these functions. Since the permutations P^S_m here are the same as those defined in [10], we adopt the following lemma from [10] which states the general relations they satisfy; a proof of this lemma can be found in the reference.

Lemma 1.

$$\begin{aligned}
P^S_{2n}P^S_{-2n} &\equiv \mathbb{1}\\
P^S_{2n+1}P^{\bar{S}}_{-(2n+1)} &\equiv \mathbb{1}\\
P^S_n &\equiv P^{\bar{S}}_{-n},
\end{aligned}$$

where $n \in \mathbb{Z}$.

Besides, the equations below are easy to derive[10]; they are listed here for possible future use.

$$\begin{aligned}
P^+_1 = P^-_{-1} &= (1\ 2)\\
P^+_{-1} = P^-_1 &= (2\ 3)\\
P^+_2 = P^-_{-2} &= (1\ 3\ 2)\\
P^+_{-2} = P^-_2 &= (1\ 2\ 3)\\
P^{\pm}_{6n+3} &\equiv (1\ 3)\\
P^{\pm}_{6n} &\equiv \mathbb{1},
\end{aligned} \tag{5}$$

where $n \in \mathbb{Z}$.

According to the graphic definitions of rotations in [3], we found that $X_l(S,m)$ and $X_r(S,n)$ have the following general algebraic forms.

$$\begin{aligned}
X_l(+,m) &= \begin{cases}(ud)^{-m/2} & \text{if } m \text{ is even,}\\ d(ud)^{(-1-m)/2} & \text{if } m \text{ is odd.}\end{cases} \quad X_l(-,m) = \begin{cases}(du)^{-m/2} & \text{if } m \text{ is even,}\\ u(du)^{(-1-m)/2} & \text{if } m \text{ is odd.}\end{cases} \\
X_r(+,n) &= \begin{cases}(ud)^{-n/2} & \text{if } n \text{ is even,}\\ (ud)^{(1-n)/2}d^{-1} & \text{if } n \text{ is odd.}\end{cases} \quad X_r(-,n) = \begin{cases}(du)^{-n/2} & \text{if } n \text{ is even,}\\ (du)^{(1-n)/2}u^{-1} & \text{if } n \text{ is odd.}\end{cases}
\end{aligned} \tag{6}$$

where $n, m \in \mathbb{Z}$. If an exponent in Eq. 7 is positive, it means, for example, $(ud)^2 = udud$. We utilize a definition in [11], which is, for a crossing sequence $X = x_1 \cdots x_i \cdots x_N,\ N \in$

$\mathbb{N}$, $X^{-1} = x_N^{-1} \cdots x_i^{-1} \cdots x_1^{-1}$. Given this, the meaning of the negative exponents in Eq. 7 is clear. For instance, $(ud)^{-2} = d^{-1}u^{-1}d^{-1}u^{-1}$.

It is obvious that the number of crossings of either $X_l(S_l, m)$ or $X_r(S_r, m)$ does not depend on the end-node state,

$$|X_l(+, m)| = |X_l(-, m)| = |X_r(+, m)| = |X_r(-, m)| = |m|, \tag{7}$$

neither does the sum of crossing numbers of $X_l(S_l, m)$ or $X_r(S_r, m)$, namely

$$\sum_{i=1}^{|m|} x_i = \sum_{i=1}^{|m|} y_i = \sum_{i=1}^{|m|} z_i = \sum_{i=1}^{|m|} w_i = -m. \tag{8}$$

where, $x_i \in X_l(+, m)$, $y_i \in X_l(-, m)$, $z_i \in X_r(+, m)$, and $w_i \in X_r(-, m)$. In addition, there is a useful relation between $X_l(S, m)$ and $X_r(S, n)$, as stated in the following Lemma.

Lemma 2. $X_l(S, m)X_r(S, -m) \equiv \mathbb{I}$. *$\mathbb{I}$ stands for no crossing.*

Proof. For m even,

$$X_l(+, m)X_r(+, -m) = (ud)^{-m/2}(ud)^{m/2} = \mathbb{I},$$
$$X_l(-, m)X_r(-, -m) = (du)^{-m/2}(du)^{m/2} = \mathbb{I};$$

for m odd,

$$X_l(+, m)X_r(+, -m) = d(ud)^{(-1-m)/2}(ud)^{(1+m)/2}d^{-1} = \mathbb{I},$$
$$X_l(-, m)X_r(-, -m) = u(du)^{(-1-m)/2}(du)^{(1+m)/2}u^{-1} = \mathbb{I}.$$

In conclusion, for any S and m, we have

$$X_l(S, m)X_r(S, -m) \equiv \mathbb{I}.$$

□

The rotation $R_{m,n}$ is actually a generalization of the simultaneous rotation $R_{n,-n}$, defined in [10] as acting on actively-interacting braids in their extremal representations, each of which is a trivial braid diagram with two identical end-node states. In this case $S_l = S_r = S$, and $X \equiv \mathbb{I}$, indicating $\sigma_X = \mathbb{1}$. Therefore, for consistency, our general rotation $R_{m,n}$ should reduce to the simultaneous rotation $R_{m,-m}$ on these braids, if we set $n = -m$. This is indeed so because

$$R_{m,-m}({}^{S}_{T_l}[T_a, T_b, T_c]^{S}_{T_r}) = {}^{(-)^m S}_{T_l+m}\,[(P^S_m(T_a, T_b, T_c))\Sigma_{X_l(S,m)\mathbb{I}X_r(S,m)}]^{(-)^m S}_{T_r-m};$$

however, Lemma 2 gives $X_l(S, m)\mathbb{I}X_r(S, m) \equiv \mathbb{I}$, such that

$$R_{m,-m}({}^{S}_{T_l}[T_a, T_b, T_c]^{S}_{T_r}) = {}^{(-)^m S}_{T_l+m}\,[P^S_m(T_a, T_b, T_c)]^{(-)^m S}_{T_r-m},$$

which is the very simultaneous rotation defined in [10].

With these ingredients, we can proceed to find out conserved quantities of a generic braid under general equivalence moves, $R_{m,n}$. One can see that unlike trivial braids under simultaneous rotations in [10], $T_l + T_r$ and the triple (T_a, T_b, T_c) are no longer conserved here for a generic rotation, only a combination of them with sum of crossing numbers, namely the effective twist $\Theta = T_l + T_r + \sum_{i=a}^{c} T_i - 2\sum_{i=1}^{|X|} x_i$ is conserved under these general equivalence moves. Besides, the conserved quantities S^2 is generalized to the **effective state** $\chi = (-)^{|X|} S_l S_r$ for generic braids. These results are summarized as the following Lemma.

Lemma 3. *Under a general rotation $R_{m,n}$, a braid's effective twist number, Θ, and its effective state, χ, are conserved.*

Proof. By Eq. 3, a general rotation $R_{m,n}$ can transform a generic braid

$$ {}^{S_l}_{T_l}[(T_a, T_b, T_c)\sigma_X]^{S_r}_{T_r}, $$

with $\Theta = T_l + T_r + \sum_{i=a}^{c} T_i - 2\sum_{i=1}^{|X|} x_i$ and $\chi = (-)^{|X|} S_l S_r$, into

$$ {}^{(-)^m S_l}_{T_l+m}[(P^{S_l}_m(T_a - m - n, T_b - m - n, T_c - m - n))\sigma_{X_l(S_l,m)XX_r(S_r,n)}]^{(-)^n S_r}_{T_r+n}, $$

with

$$ \Theta' = (T_l + m) + (T_r + n) + (\sum_{i=a}^{c} T_i - 3(m+n)) - 2(\sum_{i=1}^{|m|} y_i + \sum_{i=1}^{|X|} x_i + \sum_{i=1}^{|n|} z_i), $$

where $y_i \in X_l(S_l, m)$ and $z_i \in X_r(S_r, n)$ and

$$ \chi' = (-)^{|X_l(S_l,m)|+|X|+|X_r(S_r,n)|}(-)^m S_l (-)^n S_r. $$

Nonetheless, by Eq. 7 and Eq. 8, we obtain

$$ \Theta' = T_l + T_r + \sum_{i=a}^{c} T_i + \sum_{i=1}^{|X|} x_i + m + n - 3(m+n) - 2(-m-n) = \Theta, $$

and

$$ \chi' = (-)^{|X|+|m|+|n|}(-)^m S_l (-)^n S_r = (-)^{|X|} S_l S_r = \chi. $$

This establishes the proof. $\square$

As a direct consequence of Lemma 3, we have the following Theorem, which provides a character for actively-interacting braids.

Theorem 1. *The effective state of any actively-interacting braids is $\chi = 1$, and any braid with effective state $\chi = -1$ must be passive.*

Proof. The proof of this theorem is very simple. Any actively-interacting braid has a trivial representation, whose effective state is $\chi \equiv S^2 = 1$. Hence, according to Lemma 3, the effective state of any actively-interacting braid must be $\chi = 1$. This implies, on the other hand, any stable braid with $\chi = -1$ is never actively-interacting, and is thus passive. $\square$

All results we have obtained so far are valid for braids in any representation. Now we would like to consider the extremal representation of a braid, and try to find out how equivalence moves act on braids in this representation in particular, and the corresponding conserved quantities. The case of actively-interacting braids are investigated in [10], and it turns out that there are infinite number of extrema which are trivial braids related to each other by simultaneous rotations. Moreover, $T_l + T_r$, the triple (T_a, T_b, T_c) up to permutation, and S^2 are conserved under these simultaneous rotations. For passive braids, the situation is more involved for that their extrema are not trivial braids and that generic rotations (including generic simultaneous rotations) increase the number of crossings of an extremum. However, there are also infinite number of extrema of a passive braid due to the following Lemma.

Lemma 4. *Simultaneous rotations of the form $R_{3k,-3k}$ with $k \in \mathbb{Z}$ takes an extremum of a braid to another extremum of the braid.*

Proof. By the definition of extremal, all extrema of the same braid have the same number of crossings. Thus we only need to prove that $R_{3k,-3k},\ k \in \mathbb{Z}$ on an extremum preserves its number of crossings, then the resultant representation of braid must also be an extremum; otherwise the braid diagram undergoing the rotation should not be an extremum in the first place. Additionally, since $R_{3k,-3k} = R^{|k|}_{\pm 3,\mp 3}$ by [10], where the $\pm$ and $\mp$ depend on the sign of k. It is sufficient to prove the case of $k = \pm 1$, which are just simultaneous π rotations. We now prove that simultaneous π-rotations of a braid take the braid's crossing sequence, $X = x_1 \cdots x_i \cdots x_N,\ N \in \mathbb{N}$, to $\bar{X} = \bar{x}_1 \cdots \bar{x}_i \cdots \bar{x}_N,\ N \in \mathbb{N}$, where

$$\bar{x}_i = \begin{cases} d, & x_i = u \\ u, & x_i = d \end{cases} .$$

and thus keep the number of crossings of the braid invariant. For one-crossing braids with arbitrary S_l and S_r, it is straightforward to see that $X_l(S_l, \pm 3) x_1 X_r(S_r, \mp 3) = \bar{x}_1$, for $x_1 = u, d, u^{-1}, d^{-1}$. We assume that the this is true for any braid with up to $N \in \mathbb{N}$ crossings, with arbitrary end-node states, namely

$$X_l(S_l, \pm 3) x_1 x_2 ... x_N X_r(S_r, \mp 3) = \bar{x}_1 \bar{x}_2 ... \bar{x}_N .$$

Hence, for any braid with $N + 1$ crossings and end-node states, say S'_l and S'_r,

$$\begin{aligned} & X_l(S'_l, \pm 3) x_1 x_2 ... x_N x_{N+1} X_r(S'_r, \mp 3) \\ \xRightarrow{X_r(S,\mp 3) X_l(S,\pm 3) = \mathbb{I}} & = (X_l(S'_l, \pm 3) x_1 x_2 ... x_N X_r(S, \mp 3))(X_l(S, \pm 3) x_{N+1} X_r(S'_r, \mp 3)), \end{aligned}$$

where we inserted a pair of crossing sequences, $X_r(S, \pm 3)$ and $X_l(S, \pm 3)$ with arbitrary S, whose product is trivial by Lemma 2, between the N-th and $(N+1)$-th crossing. Note that these two crossing sequences are not created by real rotations $R_{0,\pm 3}$ or $R_{\pm 3,0}$, but rather only equal to respectively the crossing sequences created by these two rotations. Hence, according to the assumption that the claim is valid for an N-crossing braid with arbitrary end-node states, and the fact of the validity for all one-crossing braids, we arrive at

$$X_l(S_l, \pm 1)x_1x_2...x_Nx_{N+1}X_r(S_r, \mp 1) = (\bar{x}_1\bar{x}_2...\bar{x}_N)(\bar{x}_{N+1}) = \bar{x}_1\bar{x}_2...\bar{x}_{N+1}.$$

Bearing in mind that $|X| = |\bar{X}|$, therefore by induction, simultaneous rotations, $R_{\pm 3,\mp 3}$ take a generic braid with crossing sequence X to an equivalent braid with sequence $\bar{X}$, which does not change the number of crossings. This certainly indicates that $R_{3k,-3k}$ with $k \in \mathbb{Z}$ rotates an extremum to another extremum of the same braid, which validates the proof. Furthermore, by Eq. 3, Eq. 4, and Eq. 5, and with a convenient redefinition: $\bar{X} = \bar{f}(X)$, we can pin down the algebraic form of the action of a $R_{3k,-3k},\ k \in \mathbb{Z}$ on generic braids,

$$\begin{aligned} &R_{3k,-3k}({}^{S_l}_{T_l}[(T_a, T_b, T_c)\sigma_X]^{S_r}_{T_r}) \\ &= {}^{(-)^{3k}S_l}_{T_l+3k}[((1,3)^k(T_a, T_b, T_c))\sigma_{\bar{f}^k(X)}]^{(-)^{3k}S_r}_{T_r-3k}, \end{aligned} \tag{9}$$

or equivalently,

$$\begin{aligned} &R_{3k,-3k}({}^{S_l}_{T_l}[\sigma_X^{-1}(T_{a'}, T_{b'}, T_{c'})]^{S_r}_{T_r}) \\ &= {}^{(-)^{3k}S_l}_{T_l+3k}[\sigma^{-1}_{\bar{f}^k(X)}((1,3)^k(T_{a'}, T_{b'}, T_{c'}))]^{(-)^{3k}S_r}_{T_r-3k}, \end{aligned} \tag{10}$$

where $(1,3)^k$ is the permutation induced by $R_{3k,-3k}$, and $\bar{f}^k(X) = X$ for k even, while $\bar{f}^k(X) = \bar{f}(X)$ for k odd. □

Similar to the case of actively-interacting braids, there are conserved quantities under rotations in the form of $R_{3k,-3k}$, which are shown in the Lemma below.

Lemma 5. *$T_l + T_r$, $\sum\limits_{i=a}^{c} T_i$, and $\sum\limits_{i=1}^{|X|} x_i$ of an extremum of a braid are invariant under rotations of the form $R_{3k,-3k},\ k \in \mathbb{Z}$. The triple (T_a, T_b, T_c) is invariant when k is even.*

Proof. From Eq. 9 and Eq. 10, it is obvious that a rotation of the form $R_{3k,-3k},\ k \in \mathbb{Z}$ takes $T_l + T_r$ to $T_l + 3k + T_r - 3k = T_l + T_r$, turns the triple (T_a, T_b, T_c) into $(1,3)^k(T_a, T_b, T_c)$ which is again (T_a, T_b, T_c) if k is even but does not affect $\sum\limits_{i=a}^{c} T_i$ in any case, and changes $\sum\limits_{i=1}^{|X|} X_i$ to $\sum\limits_{i=1}^{|\bar{X}|} \bar{x}_i = \sum\limits_{i=1}^{|X|} x_i$. □

These conserved quantities imply that the previously defined Θ is also a conserved quantity under this class of equivalence moves, which is expected because in Lemma 3 we have shown that Θ is conserved under any rotations. That is, it is the same for any representation of a braid.

Now that we have found conserved quantities under rotations of the form $R_{3k,-3k}$, $k \in \mathbb{Z}$. The only issue left behind is that we have not proven that the simultaneous rotations of multiple of π are the only possible class of rotations under which the set of all extrema of a braid is closed. If this is true, then each conserved quantity in Lemma 5 is identical for all extrema of a braid. There are strong evidences that this is indeed the case; however, we are lack of a rigorous proof. Therefore, we only state this observation as a conjecture.

Conjecture 1. *Any rotation that transforms an extremum to another extremum of the same braid must take the form of $R_{3k,-3k}$ with $k \in \mathbb{Z}$. Assuming this, all extrema of a braid share the same $T_l + T_r$, $\sum_{i=a}^{c} T_i$, and $\sum_{i=1}^{|X|} x_i$.*

4 Algebra of interactions: symmetries and relations

In [10] it is shown that the interaction of any two active braids produces another active braid. Now that we are dealing with not only active braids but also the passive ones, one may ask what the outcome of the interaction of an active braid and a passive braid, say a propagating braid, should be. To answer this question, we need sufficient preparation, divided into the following subsections.

4.1 Conserved quantities under interactions

We first repeat in words the interaction condition formulated in [8, 10]. This condition demands that one of the two braids, say B_1 and B_2, under an interaction must be active and that the two adjacent nodes, one of B_1 and the other of B_2, are either already in or can be rotated to the configuration where they have the same state and share a twist-free edge. The latter requirement is actually the condition of a $2 \to 3$ Pachner move[8]. The algebraic form of this condition is explicitly given in [10] and is thus not duplicated here but will rather be adopted directly.

Since a braid is in fact an equivalence class of braid diagrams, a convenient choice of the representative of the class is important. Whether a braid is propagating or not is most transparent when the braid is represented by its unique representative which has no external twist. On the other hand, an active braid can always be put in a trivial representation, which simplifies the calculation of interactions. Therefore, in this section any active braid is represented by one of its extrema, i.e. a trivial diagram with external twists, and any braid which does not actively interact is represented by its unique representative.

Now when an active braid, B, meets a passive braid, B', say from the left of B' (the case where B is on the right of B' follows similarly), with the interaction condition fulfilled,

what does the resulted braid $B+B'$ look like? Here, as in [10] we use a $+$ for the operation of interaction. A special case is that B in its trivial form has no right external twist, and since B' is in the representation without external twist, one can directly apply a $2 \to 3$ move of the B's right end-node and the left end-node of B', then paly with the techniques introduced in [8] to complete the interaction. Let us address this simple case first.

Lemma 6. *Given an active braid $B = {}^{S}_{T_l}[T_a, T_b, T_c]^{S}_{T_r}$, with $T_r = 0$, and a passive braid $B' = {}^{S_l}_{0}[(T'_a, T'_b, T'_c)\sigma_X]^{S_r}_{0}$, with $S_l = S$, the interaction of B and B' with B on the left of B' produces $B'' = {}^{(-)^{T_l}S_l}_{0}[(P^{S}_{-T_l}(T_a + T'_a + T_l, T_b + T'_b + T_l, T_c + T'_c + T_l))\sigma_{X_l(S,-T_l)X}]^{S_r}_{0}$.*

Proof. As $T_r = 0$ and $S = S_l$, the interaction condition is met and thus no rotation is needed; hence, according to [8], $B + B'$ forms a connected sum of B and B', which is, in our algebraic language,

$$\begin{aligned} B + B' &= {}^{S_l}_{T_l}[T_a, T_b, T_c]^{S_l}_{0} \ \# \ {}^{S_l}_{0}[(T'_a, T'_b, T'_c)\sigma_X]^{S_r}_{0} \\ &= {}^{S_l}_{T_l}[(T_a + T'_a, T_b + T'_b, T_c + T'_c)\sigma_X]^{S_r}_{0} \\ &\cong R_{-T_l,0}\left({}^{S_l}_{T_l}[(T_a + T'_a, T_b + T'_b, T_c + T'_c)\sigma_X]^{S_r}_{0}\right) \\ &= {}^{(-)^{T_l}S_l}_{0}[(P^{S}_{-T_l}(T_a + T'_a + T_l, T_b + T'_b + T_l, T_c + T'_c + T_l))\sigma_{X_l(S,-T_l)X}]^{S_r}_{0}\ , \end{aligned} \tag{11}$$

where a rotation $R_{-T_l,0}$ is applied after the connected sum to put the resulted braid in its representative with zero external twist, which induces a permutation $P^{S}_{-T_l}$ on the left triple of internal twists, and a crossing sequence $X_l(S, -T_l)$, appended to the original X from left. ∎

However, in general the trivial diagram representing an active braid may have external twists on both external edges. If the interaction condition is satisfied when the trivial braid in this case meets a passive braid, a rotation is usually required in order to perform the connected sum algebraically for them to interact. We now deal with this.

Lemma 7. *Given an active braid $B = {}^{S}_{T_l}[T_a, T_b, T_c]^{S}_{T_r}$ on the left of a passive braid, $B' = {}^{S_l}_{0}[(T'_a, T'_b, T'_c)\sigma_X]^{S_r}_{0}$, with the interaction condition satisfied by $(-)^{T_r}S = S_l$, the interaction of B and B' results in a braid*

$$B'' = {}^{(-)^{T_l}S}_{0}[((P^{S}_{-T_l}(T_a, T_b, T_c)) + (P^{(-)^{T_r}S}_{-T_l-T_r}(T'_a, T'_b, T'_c)) + (T_l + T_r, \cdot, \cdot))\sigma_{X_l((-)^{T_r}S,-T_l-T_r)X}]^{S_r}_{0}\ ,$$

where $(T_l + T_r, \cdot, \cdot)$ is the short for $(T_l + T_r, T_l + T_r, T_l + T_r)$.

Proof.

$$
\begin{aligned}
&B+B' \\
&= {}^{S}_{T_l}[T_a,T_b,T_c]^{S}_{T_r} + {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 \\
&\cong R_{T_r,-T_r}\left({}^{S}_{T_l}[T_a,T_b,T_c]^{S}_{T_r}\right) \# {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 \\
&= {}^{(-)^{T_r}S}_{T_l+T_r}[P^S_{T_r}(T_a,T_b,T_c)]^{(-)^{T_r}S}_0 \# {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 &(12)\\
&= {}^{(-)^{T_r}S}_{T_l+T_r}[((P^S_{T_r}(T_a,T_b,T_c))+(T'_a,T'_b,T'_c))\sigma_X]^{S_r}_0 \qquad \xLeftarrow{(-)^{T_r}S=S_l} &(13)\\
&\cong R_{-T_l-T_r,0}\left({}^{(-)^{T_r}S}_{T_l+T_r}[((P^S_{T_r}(T_a,T_b,T_c))+(T'_a,T'_b,T'_c))\sigma_X]^{S_r}_0\right) \\
&= {}^{(-)^{T_l}S}_{0}[(P^{(-)^{T_r}S}_{-T_l-T_r}((P^S_{T_r}(T_a,T_b,T_c))+(T'_a,T'_b,T'_c))+(T_l+T_r,\cdot,\cdot))\sigma_{X_l((-)^{T_r}S,-T_l-T_r)X}]^{S_r}_0 \\
&= {}^{(-)^{T_l}S}_{0}[((P^S_{-T_l}(T_a,T_b,T_c))+(P^{(-)^{T_r}S}_{-T_l-T_r}(T'_a,T'_b,T'_c))+(T_l+T_r,\cdot,\cdot))\sigma_{X_l((-)^{T_r}S,-T_l-T_r)X}]^{S_r}_0 \ , &(14)
\end{aligned}
$$

where the simultaneous rotation $R_{T_r,-T_r}$, is applied to realize the interaction condition in order to do the connected sum, and the rotation $R_{-T_l-T_r,0}$ is exerted such that the final result is in the representative without external twists, which induces a permutation $P^{(-)^{T_r}S}_{-T_l-T_r}$ and a crossing sequence $X_l((-)^{T_r}S,-T_l-T_r)$ concatenated to X from left. The above equation obviously reduces to Eq. 11 when $T_r=0$. □

Nevertheless, for each active braid, there are infinite number of trivial braid diagrams which are equivalent, in the sense that any two of them are related by a simultaneous rotation $R_{n,-n}$, $n\in\mathbb{Z}$. It is then naturally to ask if the choice of the trivial braid diagram representing a braid equivalence class influences the interaction of the braid and another braid. The answer is "No". The reason is obvious because of the equivalence of the trivial diagrams. However, due to the necessity of realizing of the interaction condition in a concrete calculation of interaction, it is better to formulate this claim explicitly in our new notation as a Lemma.

Lemma 8. *For any active braid B, its interaction onto any other braid B', i.e. $B+B'$ or $B'+B$, is independent of the choice of the trivial diagram representing B.*

Proof. We prove the case of $B+B'$. Let $B_0={}^{S}_{T_l}[T_a,T_b,T_c]^{S}_{T_r}$ be a trivial diagram representing an active braid B. Let $B'={}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0$ be the passive braid on which B interacts. We assume the interaction condition is satisfied by $(-)^{T_r}S=S_l$. Any other trivial braid, say B_n, representing B can be obtained from B_0 by

$$B_n=R_{n,-n}(B_0)={}^{(-)^nS}_{T_l+n}[P^S_n(T_a,T_b,T_c)]^{(-)^nS}_{T_r-n}\ .$$

B_0+B' has already been shown in Lemma 7, but we only need Eq. 12 therein, which is the configuration of the two braid after the interaction condition is realized. If we replace

B_0 by B_n in the interaction, we have

$$\begin{aligned}
B_n + B' &= {}^{(-)^n S}_{T_l+n}[P^S_n(T_a,T_b,T_c)]^{(-)^n S}_{T_r-n} + {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 \\
&\cong R_{T_r-n,-T_r+n}\left({}^{(-)^n S}_{T_l+n}[P^S_n(T_a,T_b,T_c)]^{(-)^n S}_{T_r-n}\right) \# \, {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 \\
\xRightarrow{(-)^{2n}S=S} &= {}^{(-)^{T_r} S}_{T_l+T_r}[P^{(-)^n S}_{T_r-n} P^S_n(T_a,T_b,T_c)]^{(-)^{T_r} S}_{0} \; \# \, {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 \\
\xRightarrow{P^{(-)^n S}_{T_r-n}=P^S_{T_r}P^{(-)^n S}_{-n}} &= {}^{(-)^{T_r} S}_{T_l+T_r}[P^S_{T_r} P^{(-)^n S}_{-n} P^S_n(T_a,T_b,T_c)]^{(-)^{T_r} S}_{0} \; \# \, {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 \\
\xRightarrow{P^{(-)^n S}_{-n}P^S_n\equiv \mathbb{1} \text{ by Lemma 1}} &= {}^{(-)^{T_r} S}_{T_l+T_r}[P^S_{T_r}(T_a,T_b,T_c)]^{(-)^{T_r} S}_{0} \; \# \, {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0 ,
\end{aligned}$$

which is exactly the same as Eq. 12. That is, if B_0 interacts with B', so does B_n, and they give rise to the same result. Likewise, this is also true for the case of $B' + B$. This closes the proof. □

Now that we established Lemma 8, we may choose to always represent an active braid B by its trivial representative without right (left) external twist, in dealing with the interaction of B onto a passive braid from the left (right), which simplifies the calculation and expression because Lemma 6 directly applies. Moreover, the result of Lemma 6, namely Eq. 11, is identical to the result when the active braid is represented by its unique representative with zero external twists. This again, together with [10], shows that the choice of representative of a braid does not affect the result of the interaction involving the braid, in accordance with [8]. Examples can be found in [8], one just need to cast them in our new symbolic notation.

Equipped with this algebra, we shall prove one of our primary results.

Theorem 2. *Given an active braid, $B = {}^{S}_{T_l}[T_a,T_b,T_c]^{S}_{T_r}$, and a passive braid, $B' = {}^{S_l}_{0}[(T'_a,T'_b,T'_c)\sigma_X]^{S_r}_0$, such that $B'' = B + B'$, the effective twist number Θ is an additive conserved quantity, while the effective state χ is a multiplicative conserved quantity, namely*

$$\begin{aligned}
\Theta_{B''} &= \Theta_B + \Theta_{B'} \\
\chi_{B''} &= \chi_B \chi_{B'}.
\end{aligned} \tag{15}$$

Proof. We can readily write down

$$\Theta_B = T_l + T_r + \sum_{i=a}^{c} T_i, \quad \chi_B = 1,$$

and

$$\Theta_{B'} = \sum_{i=a}^{c} T_i - 2\sum_{j=1}^{|X|} x_j, \quad \chi_{B'} = (-)^{|X|} S_l S_r.$$

Hence, according to Eq. 14, we have

$$\begin{aligned}\Theta_{B''} &= \sum_{i=a}^{c}(T_i + T_i') + 3(T_l + T_r) - 2\sum_{j=1}^{|X_l|} x_j - 2\sum_{k=1}^{|X|} x_k \\ &= \sum_{i=a}^{c}(T_i + T_i') + (T_l + T_r) - 2\sum_{k=1}^{|X|} x_k \\ &= \Theta_B + \Theta_{B'}, \end{aligned} \tag{16}$$

where the second equality is a result of $\sum_{j=1}^{|X_l((-)^{T_r}S, -T_l-T_r)|} x_j = T_l + T_r$, by Eq. 8. Besides,

$$\chi_{B''} = (-)^{|T_l+T_r|+|X|}(-)^{T_l}SS_r = (-)^{|X|}(-)^{T_r}SS_r = (-)^{|X|}S_lS_r = \chi_{B'} = \chi_B\chi_{B'}. \tag{17}$$

□

This theorem demonstrates that the by far discovered two representative-independent conserved quantities, Θ and χ are also conserved under interactions, in the sense that the former is additive while the latter is multiplicative. This is consistent to [10] in which only interactions between active braids are discussed. In particular, χ becomes the S^2 in [10], whose conservation means that the interacting character of the braids is preserved. Furthermore, according to Theorem 1, the multiplicative conservation of χ shows that if the passive braid involved in an interaction has $\chi = -1$, the resulted braid must also has $\chi = -1$, and is thus a passive braid too.

4.2 Asymmetry between $B + B'$ and $B' + B$

It is important to note that Theorem 2 is also true for the case of interaction where the active braid is on the right of the passive braid. In fact, all the discussion above can be equally well applied to this case. One must then ask a question: does an active braid gives the same result when it interacts on to a passive braid from the left and from the right respectively? The answer is "No" in general. We now discuss this issue by considering an active braid, B, and a passive braid, B'.

First of all, even if the interaction condition is met in the case $B + B'$, there is no guarantee that the interaction condition is also satisfied in the case of $B' + B$, which means $B' + B$ is simply an impossible interaction. If we assume the interaction condition can be realized in both cases, $B + B'$ and $B' + B$ still give rise to different results, i.e. inequivalent braids, in general. Let us study the detail.

In the case of $B + B'$, by Lemma 8 we can represent B by its trivial braid diagram without T_r, viz $B \cong B_l = {}^{S}_{T_l}[T_a, T_b, T_c]^{S}_{0}$, which allows us to use Eq. 11. However, in the case of $B' + B$, we represent B by its trivial diagram without the left external twist, which is obtained by a simultaneous rotation from B_l, i.e. $B \cong B_r = R_{-T_l,T_l}(B_l) =$

${}^{(-)^{T_l}S}_{\ 0}[P^S_{-T_l}(T_a, T_b, T_c)]^{(-)^{T_l}S}_{T_l}$. Then for $B' = {}^{S_l}_{\ 0}[(T'_a, T'_b, T'_c)\sigma_X]^{S_r}_{0}$, the interaction condition in the latter case is $S_r = (-)^{T_l}S$. With also $S = S_l$, we conclude that the condition for both $B + B'$ and $B' + B$ doable is

$$S_l = (-)^{T_l} S_r. \tag{18}$$

Given this, by a similar calculation as that in Lemma 6, one can find

$$\begin{aligned} B' + B &= {}^{S_l}_{\ 0}[((T'_a + T_l, T'_b + T_l, T'_c + T_l) + (\sigma_X^{-1} P^S_{-T_l}(T_a, T_b, T_c)))\sigma_{XX_r(S_r,-T_l)}]^{(-)^{T_l}S_r}_{0} \\ &= {}^{S_l}_{\ 0}[((T'_a + T_l, T'_b + T_l, T'_c + T_l) + (\sigma_X^{-1} P^S_{-T_l}(T_a, T_b, T_c)))\sigma_{XX_r(S_r,-T_l)}]^{S_l}_{0}\ . \end{aligned} \tag{19}$$

To compare this to $B + B'$, we rewrite Eq. 11 as follows, taking Eq. 18 into account.

$$B + B' = {}^{S_r}_{\ 0}[(P^S_{-T_l}(T_a + T'_a + T_l, T_b + T'_b + T_l, T_c + T'_c + T_l))\sigma_{X_l(S_l,-T_l)X}]^{S_r}_{0} \tag{20}$$

It is important to notice that, by Eq. 19 and Eq. 20, both $B + B'$ and $B' + B$ are in the representative with zero external twists. Since we know such a representative is unique for each equivalence class of braids, there is no way for $B+B'$ and $B'+B$ to be equivalence to each other. But there are possibilities for $B + B'$ and $B' + B$ are simply equal. For this to be true, braids B and B' are pretty strongly constrained.

Firstly, one obviously has to require $S_l = S_r$ and $T_l = 2k,\ k \in \mathbb{Z}$ in Eq. 19 and Eq. 20, such that $B + B'$ and $B' + B$ have the same end-node states. This, together with the interaction condition, also indicates $S_l = S_r = S$. Keeping this in mind, one must then demand that $B + B'$ and $B' + B$ have the same crossing sequence, namely $X_l(S, -T_l)X = XX_r(S, -T_l)$, which can also be written as $X_l(S, -T_l)XX_r^{-1}(S, -T_l) = X$. With Lemma 2, this can be put in the form, $X_l(S, -T_l)XX_l(S, T_l) = X$. Since Eq. 7 reads that $X_l(S, m) = X_r(S, m)$ for m even, this condition is better expressed as

$$X_l(S, -T_l)XX_r(S, T_l) = X,$$

which now appears to be the requirement that a simultaneous $(-T_l)$-rotation leaves the crossing sequence intact. Although it has been conjectured in the end of Section 3 that only simultaneous rotations of $6k,\ k \in \mathbb{Z}$ are able to achieve this condition, we would like to keep its current general format because the conjectured has not been proved. However, interestingly, we will see an automatic input of $6k,\ k \in \mathbb{Z}$ shortly.

From this point on, three possibilities will arise, each of which leads to a different condition on top of the conditions found above. The first possibility is that we do not put any constraint on the internal twists. Hence, according to Eq. 19 and Eq. 20, such that $B'+B$ equals $B+B'$ we have to demand $\sigma_X^{-1}P^S_{-T_l} = P^S_{-T_l} \equiv \mathbb{1}$. An immediate consequence is that $\sigma_X = \mathbb{1}$, which however does not constraint the pattern of X very much. Moreover, as we know that T_l is even, then by Eq. 5, $P^S_{-T_l} \equiv \mathbb{1}$ if and only if $T_l = 6j,\ j \in \mathbb{Z}$. With this T_l, Lemma 4 ensures the fulfillment of the condition, $X_l(S, -T_l)XX_r(S, T_l) = X$, which now appears to be redundant in this case.

The second possibility is to relax the first one a little bit by only requiring $\sigma_X^{-1}P^S_{-T_l} = P^S_{-T_l}$ which only means that $\sigma_X = \mathbb{1}$ but no criteria for $P^S_{-T_l}$. Then we notice that while

in Eq. 20 both of the triples (T_a, T_b, T_c) and (T'_a, T'_b, T'_c) are under the same permutation $P^S_{-T_l}$, in Eq. 19 only the triple (T_a, T_b, T_c) is permuted by $P^S_{-T_l}$. Therefore, the only way to make $B + B' = B' + B$ is to mandate $P^S_{-T_l}(T'_a, T'_b, T'_c) = (T'_a, T'_b, T'_c)$. Note that this does not necessarily mean $P^S_{-T_l} = \mathbb{1}$, e.g. when $P^S_{-T_l} = (1,\ 2)$ and $T'_a = T'_b$.

The last possibility is to remove even the condition $\sigma_X = \mathbb{1}$. As a result, the constraint on internal twists turns out to be stronger than those in the previous two possibilities. It is not hard to see, we must have in general $T_a = T_b = T_c$ but still $P^S_{-T_l}(T'_a, T'_b, T'_c) = (T'_a, T'_b, T'_c)$ in the meanwhile.

We have now exhausted all conditions with the most general consideration. More importantly, although we restricted our discussion to the case where B' is a passive braid, the conditions automatically applies to the case where B' is even an active braid in its unique representation. Let us summarize this in the following theorem as another primary result of this paper.

Theorem 3. *Given an active braid $B = {}^S_{T_l}[T_a, T_b, T_c]^S_0$, and an arbitrary braid (passive or active) in its unique representative, namely $B' = {}^{S_l}_0[(T'_a, T'_b, T'_c)\sigma_X]^{S_r}_0$, active or passive, for $B + B' = B' + B$ to be true, we demand*

$$\begin{aligned} &S_l = S_r = S \\ &T_l = 2k,\ k \in \mathbb{Z} \end{aligned} \tag{21}$$

and

$$X_l(S, -T_l) X X_r(S, T_l) = X \tag{22}$$

and any of the following three:

1.

$$\sigma_X = \mathbb{1} \tag{23}$$

$$T_l = 6k,\ k \in \mathbb{Z} \tag{24}$$

2.

$$\begin{aligned} &\sigma_X = \mathbb{1} \\ &P^S_{-T_l}(T'_a, T'_b, T'_c) = (T'_a, T'_b, T'_c) \end{aligned} \tag{25}$$

3.

$$\begin{aligned} &T_a = T_b = T_c \\ &P^S_{-T_l}(T'_a, T'_b, T'_c) = (T'_a, T'_b, T'_c). \end{aligned} \tag{26}$$

An important remark is that Theorem 3 is based on the assumption that Conjecture 1 may be incorrect. If Conjecture 1 happen to be correct (there is a strong evidence that it is indeed so), then the conditions in Theorem 3 should be modified as follows. The satisfaction of Eq. 22 indicates $T_l = 6k,\ k \in \mathbb{Z}$, which immediately ensures that Eq. 24

holds and that $P^S_{-T_l} \equiv \mathbb{1}$. Therefore, by simple logic the condition that $B + B' = B' + B$, if Conjecture 1 stands, is reduced to:

$$\begin{aligned} S_l &= S_r = S \\ T_l &= 6k,\ k \in \mathbb{Z} \end{aligned} \tag{27}$$

and either

$$\sigma_X = \mathbb{1} \tag{28}$$

or

$$T_a = T_b = T_c. \tag{29}$$

The discussion above focuses on how $B + B'$ is equal to $B' + B$. Nevertheless, since [11] discovered discrete transformations of braids, which are mapped to $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$, and their products, and discussed their actions on braid interactions, $B + B'$ and $B' + B$ may be just related by a discrete transformation. As pointed out in [11], the transformations $\mathcal{P}$, $\mathcal{T}$, $\mathcal{CP}$, and $\mathcal{CT}$ swap the two braids undergoing an interaction. That is, for example, $\mathcal{P}(B + B') = \mathcal{P}(B') + \mathcal{P}(B)$. If B and B' happen to be invariant under $\mathcal{P}$, $B' + B$ is then equal to $\mathcal{P}(B + B')$. We will not involve the detailed mathematical formats of these transformations, which can be found in [11]; rather, we list below the conditions for this to be true.

$$B' + B = \begin{cases} \mathcal{P}(B + B'), & B = \mathcal{P}(B),\ B' = \mathcal{P}(B') \\ \mathcal{T}(B + B'), & B = \mathcal{T}(B),\ B' = \mathcal{T}(B') \\ \mathcal{CP}(B + B'), & B = \mathcal{CP}(B),\ B' = \mathcal{CP}(B') \\ \mathcal{CT}(B + B'), & B = \mathcal{CT}(B),\ B' = \mathcal{CT}(B'). \end{cases} \tag{30}$$

4.3 The algebraic structure

With the help of this notation and the algebraic method established on it we are able to show that the set of all stable braids, namely the active braids, propagating braids, and stationary braids, form an algebra under the braid interaction. This algebra is closed. The reason is that any interaction of the type defined in [8] of two stable braids never leads to an instable braid due to the stability condition put forward in [8, 14].

In [10] it is demonstrated that an interaction between two active braids always results in another active braid. On the other hand, interactions between active and passive braids turn out to be more complicated and involved. However, provided with all the discussion in previous sections, we can try to answer the question raised at the beginning of Section 4. This question can be first partly answered by the following Theorem.

Theorem 4. *The resulted braid of any interaction between an active braid and a passive braid is again passive.*

Proof. Recalling Theorem 1 and Theorem 2, since the effective state χ is a multiplicative conserved quantity under interaction and an active braid must have $\chi = 1$, the interaction of an active braid and any passive braid with $\chi = -1$ must leads to a passive braid with $\chi = -1$.

However, this is not a complete proof because a passive braid may also have $\chi = 1$. A full proof can be easily constructed. by contradiction. For this purpose, we need the following facts, extracted from [8], of passive braids:

1. A stationary braid is neither left nor right completely reducible.
2. A (left-) right-propagating braid is never completely-reducible from its (right) left end-node.
3. A two-way propagating braid is not completely-reducible from either end-node; otherwise it must be both left and right completely-reducible, which makes it an active braid if equipped with appropriate twists.

Now let us consider an active braid B in an arbitrary trivial representative and a passive braid B' in its unique representative, with the interaction condition met, their interaction, say $B + B'$ (the case of $B' + B$, if possible, will follow similarly), results in, by Eq. 13, with internal twists ignored because they are irrelevant,

$$B + B' = {}_{T_l+T_r}^{\quad S_l}[(\cdot,\cdot,\cdot)\sigma_X]_0^{S_r} \tag{31}$$

We do not apply a rotation of the left end-node of $B + B'$ because things will become less transparent otherwise. Note that according to Eq. 31, at this stage, the two end-nodes of $B + B'$ are in the same states respectively as those of B' before the interaction, and also that it has the same crossing sequence X as B' does. This means the irreducibility of $B + B'$ respects that of B'.

We know that an active braid must be both left and right completely-reducible. Now that B' is passive, it is never completely-reducible from both sides, which means $B + B'$ is not either by Eq. 31. Otherwise, B' should be two-way completely reducible in the first place, which is contradictory to any basic facts listed above of a passive braid. Therefore, the theorem holds. □

Since an interaction of two active braids gives rise to active braids only, as aforementioned, Theorem 4 then immediately entitles the set of active braids a subalgebra of the algebra of stable braids.

We still need to discuss if the interaction of an active braid and a passive propagating (stationary) braid creates a propagating (stationary) braid. The answer is "Not always". To illustrate this, we show two examples.

Let us consider an active braid, $B = {}_{-1}^{+}[1, 1, -1]_0^{+}$, and a left-propagating braid, $B_p = {}_0^{+}[(-5, -5, 1)\sigma_{ud^{-1}}]_0^{+}$, whose graphical presentations can be found in [8, 11]. Hence, by Eq.

11 we have

$$
\begin{aligned}
B + B_p = & \; {}^{-}_{0}[(P_1^+(-5,-5,-1))\sigma_{u^{-1}ud^{-1}}]^{+}_{0} \\
\xrightarrow{P_1^+=(1,\,2)} = & \; {}^{-}_{0}[(-5,-5,-1)\sigma_{d^{-1}}]^{+}_{0} \;,
\end{aligned} \tag{32}
$$

which is an irreducible braid according to [3], and is thus stationary. This example shows the interaction of an active braid and a propagating braid can result in a stationary braid. The reason for such a situation to arise is due to the pair cancelation of crossings and the change of the end-node state, as a consequence of the interaction, which is lucid in the example above.

On the other hand, An active braid and a stationary can also produce a propagating braid via their interaction. For this sake, we can use the braid in Eq. 32 as the stationary one and name it B_s, and consider an active braid $B = {}^{-}_{1}[-1,-1,1]^{-}_{0}$. Also by Eq. 11 we obtain

$$
\begin{aligned}
B + B_s = & \; {}^{+}_{0}[(P_{-1}^-(-5,-5,1))\sigma_{ud^{-1}}]^{+}_{0} \\
\xrightarrow{P_{-1}^-=(1,\,2)} = & \; {}^{+}_{0}[(-5,-5,1)\sigma_{ud^{-1}}]^{+}_{0} \;,
\end{aligned} \tag{33}
$$

which is the very B_p in the previous example.

Above all, the set of all stable braids form an algebra with interaction as the binary operation, which is associative, of the algebra. Stable braids are local excitations of embedded spin networks which are considered to be basis states describing the fundamental space-time. Consequently, a physical state is usually a superposition of these basis states. It is therefore clear that braid interaction, as the binary operation of the Algebra of stable braids, is bilinear. Within this algebra, the set of active braids behaves as a subalgebra. In addition, due to the asymmetry between left and right interactions elaborated in Section 4.2, this algebra is noncommutative.

5 Conclusion, discussion and outlook

Conservation laws play a pivotal role in revealing the underlying structure of a physical theory. By means of invariants and conserved quantities we are able to determine how the content of the theory relates to particle physics, or what kind of new mathematical and/or physical inputs are necessary so that the theory is meaningful.

We have generalized the algebraic notation of active braids, proposed in [10], to all our braids, found a set of equivalence relations relating them, and developed conserved quantities associated with these relations. More importantly, by means of this notation we studied the interaction between active braids and passive braids, which leads us to the fact that the set of all stable braids forms a noncommutative algebra, with a subalgebra containing only active braids. From this we have found both additive and multiplicative conserved quantities of braids under interaction. These are not only dynamically conserved but also conserved under the equivalence moves.

A possible next step is to determine which of these conserved quantities may correspond to quantum numbers, together with the results for interactions of braids found in this paper, to fully classify the set of braids. These conserved quantities find an application in a companion paper[11] of us, in which discrete transformations of our braids have been discovered and mapped to charge conjugation, parity, time reversal and their products. The results of interactions also stimulate another work in this direction[12].

The ultimate physical content of our braids cannot be fully understood at this stage. In [13] regarding the braids of 3-valent embedded spin networks, a tentative mapping between the 3-valent braids and Standard Model particles is proposed, with however the absence of dynamics. In the 4-valent case, as also discussed in the companion paper[11], such a direct mapping, if not impossible, is at least obscure at this level of understanding of the braids. A reason is that the dynamics, namely the propagation and interaction of 4-valent braids, strongly constraints the possible set of twists, crossing sequence, and ene-node states of a braid for it to be propagating and/or interacting. In addition, the closed form of this constraint is still missing. Consequently, one is not supposed to assign a 4-valent braid any topological property just for it to be possibly a Standard Model particle, which is what has been done in the 3-valent case. More study and in particular maybe new mathematical tools are needed to reveal whether the 4-valent can directly correspond to Standard Model particles.

If our 4-valent braids are more fundamental entities than the Standard Model particles are, then what do they correspond to, what do their interactions mean actually, and how do they give rise to Standard Model particles under certain semi-classical limit? These are profound but natural questions to ask. However, our current understanding of 4-valent braids has not provided sufficient knowledge to give an answer. The realm of braids of spin networks is enormous, and a great deal of future work must be done.

For example, in our study we have yet not included spin network labels which are normally representations of gauge groups, or of the quantum groups of the corresponding gauge groups. This may make one misunderstand that the properties of braids are independent of the spin network they live on. This is nevertheless not true. On the one hand, although that a braid is propagating and/or interacting depends on its topological setting, whether it can indeed propagate away from its location and/or interact with its adjacent braids depends on the structure of its neighborhood and hence of the whole spin network it is on. On the other hand, when spin network labels are taken into account, a braid becomes manifestly dependent of its spin network, with only its topological properties unchanged. Braids of the same topology but different set of spin network labels would be considered physically different, though maybe not different particles.

Moreover, with spin network labels, a dynamical move, e.g. a $2 \to 3$ move, may have a superposition of outcomes in identical topological configuration but different set of spin network labels; each outcome has a certain probability amplitude. However, the original set of topological quantities of a braid which is essential for the braid to be propagating and interacting is still valid even after spin network labels are considered.

We know that the very notion of particles in local quantum field theories depends on

the background geometry of the theories. Our braids also depend on the spin network they live on. In spite of the fact that how to take a physically meaningful semi-classical limit of our approach remains an open question, the matter particles resulting from the braids in a semi-classical background as a reasonable such limit of superposed spin networks would be expected to depend on the background geometry as well.

It is therefore very interesting and necessary to study the effects of spin networks in our future research. Our companion paper[11] also indicates, from another perspective, the necessity of taking into account spin network labels.

Acknowledgements

We are grateful to Yong-Shi Wu for his helpful discussion, especially his stimulation of a very interesting future direction along this research line. YW is indebted to his Advisor, Lee Smolin, for his encouragement and critical comments. Thanks also go to Jonathan Hackett for discussions. SH is also grateful to the hospitality of the Perimeter Institute. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. Research at Peking University is supported by NSFC (nos.10235040 and 10421003).

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